Perspectives on Spectra

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Abstract

These are notes for three lectures given at the Young Topologists Meeting (as one of the two invited speakers/"old topologists"). They are not the most polished document, but should give an overview of what was discussed in the lectures. They also don't exactly reflect the structure of the talks as they actually happened: while I originally planned Lecture n to cover exactly Section n, over the course of the lectures I realized that particularly Lecture 2 was planned a bit too ambitiously, and that it would be more beneficial to everyone to cover that in more detail over the course of two lectures, and leave Section 3 out altogether. I still decided to leave it in the notes for everyone who was particularly curious about Lecture 3.

1 Talk 1: Spectra

1.1 The ∞-category of spectra

What are spectra? They grew out of attempts to study two seemingly different types of objects:

- generalized cohomology theories
- "spaces up to stabilisation", i.e. as some objects to study what happens under $\Sigma : \pi_n(X) \to \pi_{n+1}(\Sigma X)$ as $n \to \infty$.

The first one directly leads one to consider sequences of pointed spaces E_n with $E_n \simeq \Omega E_{n+1}$, these represent a generalized cohomology theory via

$$\widetilde{E}^n(X) \simeq [X, E_n],$$

where the equivalence $E_n \simeq \Omega E_{n+1}$ corresponds to the suspension isomorphism in the cohomology theory. By Brown representability, in fact each generalized cohomology theory is of this form.

Definition 1.1. A spectrum is a sequence of pointed spaces E_n together with equivalences $E_n \xrightarrow{\simeq} \Omega E_{n+1}$.

Of course, we want to organize these into a category: A map of spectra should be a sequence of $f_n : E_n \to F_n$ compatible with the equivalences $E_n \to \Omega E_{n+1}$ and $F_n \to \Omega F_{n+1}$, and here to get the right mapping spaces, the compatibility should be up to chosen homotopies. So a map should be a sequence of maps f_n and a sequence of homotopies H_n making these squares commute. If we want to turn this definition into an ordinary (or even space-enriched) category, we'll have lots of issues: Composing morphisms involves composition of homotopies, and that is typically not associative. However, the problem goes away in ∞-category theory:

Definition 1.2. Sp, the ∞-category of spectra is the full subcategory of

$$\mathrm{Fun}(N(\mathbb{Z} \times \mathbb{Z}), \mathcal{S}_*)$$

on those functors which take the off-diagonal entries to a contractible space, and on each of the squares formed by (n, n), $(n, n+1)$, $(n+1, n)$, $(n+1, n+1)$, the induced diagram

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & X_{n+1} \end{tikzcd}$$

induces an equivalence $X_n \to \Omega X_{n+1}$.

Here, the data encoded by such a diagram can still be seen to be exactly the spaces X_n along the diagonal as well as the equivalences $X_n \to \Omega X_{n+1}$ encoded by the squares along the diagonal, and everything else is somehow encoded by composition. In fact, we could have replaced $N(\mathbb{Z} \times \mathbb{Z})$ by the (less elegant) simplicial subset formed by just those squares, glued end-to-end, and would get the same category. The problems with composition are neatly absorbed by the fact that in an ∞-category in general, "composition" is not a uniquely chosen operation, but instead characterized itself by a kind of universal property (the horn-filling). Composition is of course still associative up to homotopy (in fact, much more coherently associative), so we don't lose anything this way.

If one is willing to have the category Sp itself only be characterized by a universal property, one can be even slicker, and characterize Sp as

$$\mathrm{Sp} \simeq \lim_{\mathbb{Z}}(\ldots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \ldots)$$

as a limit in $\widehat{\mathrm{Cat}}_\infty$, the category of (possibly big) ∞-categories. You can think of an object in such a limit as a family of coherently compatible objects in each of the categories in the diagram, which in this case unwinds exactly to a sequence of objects X_n with equivalences $X_n \simeq \Omega X_{n+1}$.

1.2 Spectra as stabilisation of spaces

In a way the definition of spectra above was biased towards their role in Brown representability. The above diagram is one of categories and right adjoint functors, so there should be some relationship to the corresponding diagram of left adjoints

$$\ldots \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \ldots.$$

Indeed, one has the following:

Proposition 1.3.

$$\mathrm{Sp} \simeq \mathrm{colim}_{\mathbb{Z}}^{\mathrm{Pr}^L}(\ldots \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \mathcal{S}_* \xrightarrow{\Sigma} \ldots),$$

with the colimit computed in the category of presentable ∞-categories with left adjoint functors as morphisms.

Here, presentable categories are loosely speaking categories which have all small colimits, and are generated under those by a set of small enough (κ-compact) objects. The above statement is a combination of two facts: Passing to right adjoints gives a contravariant equivalence $\mathrm{Pr}^L \simeq (\mathrm{Pr}^R)^{\mathrm{op}}$, and in both of those categories, limits are just formed underlying, i.e. preserved by the forgetful functor to $\widehat{\mathrm{Cat}}_\infty$.

So Sp arises from $\mathcal{S}_*$ by stabilizing under Σ, we just have to do it in a way that also respects colimits (the structure we have on objects of Pr^L). The canonical functor $\mathcal{S}_* \to \mathrm{Sp}$ coming from this colimit description is left adjoint to the forgetful functor $\mathrm{Sp} \to \mathcal{S}_*$ coming from the limit description, we write those Σ^∞, Ω^∞. $\Sigma^\infty S^0$ is the sphere spectrum $\mathbb{S}$.

In fact, $\mathcal{S}_* = \mathrm{Ind}(\mathcal{S}_*^\omega)$ is compactly generated, and one can obtain from this that $\mathrm{Sp} \simeq \mathrm{Ind}(\mathrm{Sp}^\omega)$ and $\mathrm{Sp}^\omega \simeq \mathrm{colim}(\ldots \xrightarrow{\Sigma} \mathcal{S}_*^\omega \xrightarrow{\Sigma} \mathcal{S}_*^\omega \xrightarrow{\Sigma})$, now really a colimit in Cat_∞. So, objects of Sp^ω come from some finite stage $X \in \mathcal{S}_*^\omega$, as $\Sigma^{\infty-n}X$, and we have that

$$\mathrm{Map}_{\mathrm{Sp}^\omega}(\Sigma^{\infty-n}X, \Sigma^{\infty-n}Y) \simeq \mathrm{colim}_k \, \mathrm{Map}_{\mathcal{S}_*}(\Sigma^k X, \Sigma^k Y).$$

This description of compact spectra Sp^ω is sometimes called the Spanier-Whitehead category.

1.3 Stabilisation and modes of Pr^L

The description of Sp as colimit in Pr^L does not only give a satisfying perspective related to "stabilisation of spaces", it also can be used to explain the structure of Sp as symmetric-monoidal category, and give a nice perspective on stabilisation in general. To explain this, we need the following:

Theorem 1.4 (Lurie)**.** Pr^L *admits a symmetric-monoidal structure, with $\mathcal{C} \otimes \mathcal{D}$ characterized by the following universal property:*

Left-adjoint (i.e. colimit-preserving) functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ are the same as functors

$$\mathcal{C} \times \mathcal{D} \to \mathcal{E}$$

which are "bi-colimit preserving": They preserve colimits in each variable similarly. Its unit is $\mathcal{S}$, the category of spaces.

Now, if a category $\mathcal{C}$ has a zero object, we can talk about suspension, with $\Sigma_{\mathcal{C}} X := 0 \amalg_X 0$. On Sp, this defines a self-equivalence (basically because we inverted Σ by forming the colimit). Now, for $\mathcal{C} \in \mathrm{Pr}^L$ with a zero object,

$$\mathrm{Sp} \otimes \mathcal{C} \simeq \mathrm{colim}_{\mathbb{Z}}(\ldots \xrightarrow{\Sigma \otimes \mathrm{id}_{\mathcal{C}}} \mathcal{S}_* \otimes \mathcal{C} \xrightarrow{\Sigma \otimes \mathrm{id}_{\mathcal{C}}} \ldots) \simeq \mathrm{colim}_{\mathbb{Z}}(\ldots \xrightarrow{\Sigma_{\mathcal{C}}} \mathcal{C} \xrightarrow{\Sigma_{\mathcal{C}}} \ldots),$$

i.e. the "stabilisation" of $\mathcal{C}$. If $\Sigma_{\mathcal{C}}$ is already an equivalence ($\mathcal{C}$ is stable), then this is $\mathcal{C}$ itself. In particular, $\mathrm{Sp} \otimes \mathrm{Sp} \simeq \mathrm{Sp}$, and this equivalence in particular gives rise to a "bi-colimit preserving functor" $\mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. This is the tensor product of spectra, and if one says things in a more structured way gives rise to a symmetric monoidal structure. It is uniquely characterized by being compatible with colimits and having $\mathbb{S}$ as unit.

A nice way to say this is that since $\mathrm{Sp} \otimes \mathcal{C} \simeq \mathcal{C}$ if and only if $\mathcal{C}$ has a zero object and $\Sigma_{\mathcal{C}}$ is an equivalence, the full subcategory $\mathrm{Pr}^L_{\mathrm{st}} \subseteq \mathrm{Pr}^L$ on those $\mathcal{C}$ with $\mathrm{Sp} \otimes \mathcal{C} \simeq \mathcal{C}$ is closed under the Lurie tensor product, and itself symmetric-monoidal with unit Sp. In fact, it is also $\mathrm{Mod}_{\mathrm{Sp}}(\mathrm{Pr}^L)$, but because it is a full subcategory, the module structure is really just a property. One has a left adjoint $\mathrm{Pr}^L \to \mathrm{Pr}^L_{\mathrm{st}}$ to the forgetful functor, given by tensoring with Sp, which by the above can be viewed as stabilisation. As $\mathrm{Sp} \otimes \mathcal{S} \simeq \mathrm{Sp}$, we may also interpret Sp as the initial presentable stable category under $\mathcal{S}$. As $\mathcal{S}$ admits a description as free presentable category on one object, Sp can be interpreted as free presentable stable category on one object. (Alternatively, this is the universal property $\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{D}) \simeq \mathcal{D}$ for $\mathcal{D}$ presentable stable.)

As an aside comment, these "idempotent algebras" in Pr^L, i.e. categories $\mathcal{C}$ with a morphism $\mathcal{S} \to \mathcal{C}$ inducing equivalences $\mathcal{C} \otimes \mathcal{C} \simeq \mathcal{C}$, are a bit rare. They have been called "modes of Pr^L" by Schlank and collaborators. Sp is one example, $\mathcal{S}_*$ another (its modules correspond to categories with 0-object), and for example Set is yet another (its modules are 1-categories). There are other flavors of modes related to chromatic localisation and semiadditivity, but it is an interesting question if there are previously undiscovered further ones.

1.4 But aren't these "Ω-spectra"...?

Historically, people constructed a large number of different model categories of spectra. As we saw above, the ∞-category Sp we wrote down, with objects given by sequences of spaces with chosen deloopings, isn't hard to write down, but would be harder to write down as something like a space-enriched 1-category, because of troubles with nonassociativity of composition of homotopies. There is really only one correct ∞-category of spectra, which we saw above is characterized by an elegant universal property, and the various 1-categorical constructions can be thought of as different ways to attempt to approximate this by a 1-category, possibly strictifying some parts of the structure.

The common theme is that all these other categories admit a functor to spectra. For example, given a sequence of spaces X_n with maps $\Sigma X_n \to X_{n+1}$ (without any equivalence condition), we can build an actual spectrum as $\mathrm{colim}_n \Sigma^{\infty - n} X_n$. In the model category of spectra called "sequential spectra", this is hidden in the fibrant replacement of some model structure, but in terms of the ∞-category Sp we can cleanly describe this simply in terms of colimits and $\Sigma^{\infty-n}$. We might call the sequence of $\Sigma X_n \to X_{n+1}$ actually a sequential prespectrum, and taking the colimit above allows us to produce an actual spectrum. For other models of spectra, like symmetric or synthetic spectra, there are similar colimit constructions that get us from some 1-categorical encoding of (pre)spectra to actual spectra. In the cases where one has a suitable monoidal structure on the 1-category (e.g. symmetric/orthogonal prespectra), these functors are symmetric-monoidal.

The often-praised computational benefits of a model-categoric presentation of spectra can still be accessed this way: If a specific spectrum or map of spectra can be easily described in terms of an explicit model in one of these stricter categories, we may still use that, by simply applying the functor to Sp. The fact that one can go back (statements of the form that every spectrum can be represented by, say, an orthogonal prespectrum, i.e. that these are really a model for Sp), is not really needed: Usually, it is much better to stay in the ∞-category Sp, where one has strong universal properties (for example, uniquely characterizing the monoidal structure).

So the point of view is that there is really just one ∞-category of spectra, with the various historical models just being different ways of presenting spectra. (Not unlike the relationship between groups and group presentations.)

2 Talk 2/3: Filtered and synthetic spectra

(This lecture in particular is based on a course I taught in Oslo last semester. For much more details, consider the notes, available at `https://akrause.xyz/files/nilpotence.pdf`)

How can we understand spectra? One of the ways we talk about spectra is by their homotopy groups:

$$\pi_n(X) = [\mathbb{S}^n, X]$$

with $\mathbb{S}^n = \Sigma^n\mathbb{S}$, which makes sense for $n \in \mathbb{Z}$ since Σ_{Sp} is invertible.

Homotopy groups are conservative: If $X \to Y$ induces isomorphisms on all π_n, it is actually an equivalence. But the map is necessary: If X and Y have abstractly isomorphic homotopy groups, it is not true that they are equivalent. So a spectrum is more than the collection of its homotopy groups. What more though?

Definition 2.1. A filtered spectrum is a sequence

$$\to X_n \to X_{n-1} \to \dots$$

of spectra, indexed by $\mathbb{Z}^{\mathrm{op}}$. More formally, we have a category

$$\mathrm{FilSp} = \mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}, \mathrm{Sp}).$$

We also have a category $\mathrm{gr}\,\mathrm{Sp} = \prod_{\mathbb{Z}} \mathrm{Sp}$, and a functor $\mathrm{gr} : \mathrm{FilSp} \to \mathrm{gr}\,\mathrm{Sp}$ which takes $X_\bullet$ to the graded spectrum whose degree n part is X_n/X_{n+1} (the cofiber of $X_{n+1} \to X_n$).

One specific way to produce filtered spectra is to start with an ordinary spectrum X and take its Whitehead filtration $\tau_{\geq *}X$. Here $\tau_{\geq n}X$ is a spectrum which has vanishing homotopy groups below degree n and a map $\tau_{\geq n}X \to X$ inducing isos on π_k for $k \geq n$. This is unique up to equivalence and functorial in X, by the theory of t-structures. The n-th associated graded term is the cofiber of $\tau_{\geq n+1}X \to \tau_{\geq n}X$, which has a single homotopy group in degree n, given by $\pi_n(X)$. This can be seen to uniquely characterize it, as $\Sigma^n H\pi_n(X)$, an Eilenberg-MacLane spectrum.

Thus, the homotopy groups of X tell us precisely what the associated graded of the Whitehead filtration of X is. What extra data do we need to specify on the associated graded to recover a filtered spectrum? This turns out to have a nice answer:

Theorem 2.2 (Ariotta)**.** *The associated graded refines to an equivalence*

$$\widehat{\mathrm{FilSp}} \to \mathrm{Ch}(\mathrm{Sp})$$

where the target category is a category of "coherent cochain complexes".

Here, a coherent cochain complex consists of spectra C^n, maps $\partial^n : C^n \to C^{n+1}$, but then instead of simply the condition $\partial^{n+1} \circ \partial^n \simeq 0$, it consists further of chosen nullhomotopies for those composites, chosen nullhomotopies for Toda brackets $\langle \partial^{n+2}, \partial^{n+1}, \partial^n \rangle$ formed with these nullhomotopies, etc. One can make this precise in terms of a functor category.

In the special case of the Whitehead filtration (correctly accounting for the shifts in Ariotta's equivalence), the additional structure takes the form of "connecting homomorphism maps" $\Sigma^{2n} H\pi_n(X) \to \Sigma^{2n+2} H\pi_{n+1}(X)$, nullhomotopies for their composites, etc. Even though the Eilenberg-MacLane spectra here are "concentrated in one degree", the respective mapping spaces are highly nontrivial. For example,

$$[\Sigma^{2n} H\pi_n(X), \Sigma^{2n+2} H\pi_{n+1}(X)] \cong [H\pi_n(X), \Sigma^2 H\pi_{n+1}(X)] \cong H^2(H\pi_n(X); \pi_{n+1}(X)),$$

which is generally nontrivial. If one performs the same constructions in the derived category $\mathcal{D}(R)$ of a ring, and spells out what the additional data is which is needed to piece together a complex from its homology groups (R-modules), the groups in which these coherent cochain complex differentials land are Ext-groups over R. Here, similarly, the homotopy groups are just abelian groups, but the information required to piece them together to a spectrum again lands in "Ext-groups" different from the ones in $\mathcal{D}(\mathbb{Z})$. This is usually not a tractable way to describe a spectrum, but it is a nice philosophy, and tells us to expect Sp to be completely controlled by abelian groups and the cohomology of Eilenberg-MacLane spectra.

In general, filtrations on spectra provide a way to cut a spectrum up into individual pieces which we might understand better. The main mechanism to pass from information about the associated graded to information about the "underlying spectrum" $X_{-\infty} := \mathrm{colim}_n X_n$ is a spectral sequence

$$\pi_n(\mathrm{gr}_i X) \Rightarrow \pi_n(X_{-\infty}).$$

We will display these according to the following grading convention: The coordinates are (n, s) with $s = i - n$, so that the level lines of i are the codiagonals. We call the first page the E_2 page. Differentials on the E_r page go one to the left and r up.

We can construct interesting filtrations from the Whitehead filtration. One extremely important example comes from the following:

Theorem 2.3 (Adams)**.** *Let X be a bounded below p-complete spectrum. Then*

$$X \simeq \lim_{\Delta} \left(\; H\mathbb{F}_p \otimes X \rightrightarrows H\mathbb{F}_p \otimes H\mathbb{F}_p \otimes X \dots . \; \right)$$

We get an interesting filtration on such X by setting

$$F^{\geq i} X = \lim_{\Delta} \left(\; \tau_{\geq i}(H\mathbb{F}_p \otimes X) \rightrightarrows \tau_{\geq i}(H\mathbb{F}_p \otimes H\mathbb{F}_p \otimes X) \dots . \; \right)$$

Here, the i-th associated graded still is a limit of spectra with a single homotopy group $\pi_i(H\mathbb{F}_p^{\otimes \bullet + 1} X)$ each. These can be described in terms of $H_*(X; \mathbb{F}_p)$ and the dual Steenrod algebra $\mathcal{A}_* = \pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$. One obtains:

Theorem 2.4 (The Adams spectral sequence)**.** *For X bounded below and p-complete, there is a spectral sequence*

$$\mathrm{Ext}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(X; \mathbb{F}_p)) \Rightarrow \pi_*(X).$$

If we draw a picture of the Adams spectral sequence, say for the 2-complete sphere $\mathbb{S}_2^\wedge$, we can read off the associated graded of a filtration on the homotopy groups along columns (after differentials), but we can actually see more from the picture. We can also tell what the stages of the filtration $F^{\geq i}\mathbb{S}_2^\wedge$ are: They are spectra whose homotopy groups are computed by a truncated version of the Adams spectral sequence, where we cut off the part $n + s < i$. Similarly, $F^{[i,j]}\mathbb{S}_2^\wedge = F^{\geq i}/F^{\geq j+1}$ is computed by a truncated version where we removed everything outside the range $n + s \in [i, j]$. We see how the associated graded gives us exactly the diagonals in the E_2 page, and how the longer truncations $F^{[i,i+r-1]}\mathbb{S}_2^\wedge$ "know" about the d_r differentials.

Formally, the statement that allows us to perform the above constructions is that we have "descent for bounded below p-complete modules" along $\mathbb{S} \to H\mathbb{F}_p$, allowing us to recover X from the base-change $H\mathbb{F}_p \otimes_{\mathbb{S}} X$ and "descent data" in the form of the rest of the diagram.

We can do this for other ring spectra replacing $H\mathbb{F}_p$. One great choice is MU. This is a spectrum whose homotopy groups $\pi_*(MU)$ and homology groups $H_*(MU)$ are both given by polynomial rings over $\mathbb{Z}$ (both of the same form, with one generator in each even degree, but the map $\pi_*(MU) \to H_*(MU)$ is a subtle injective map).

Descent along $\mathbb{S} \to MU$ even works for bounded below spectra (no completeness assumption). So, for any bounded below spectrum, we again get a filtration

$$F^{\geq i} X = \lim_{\Delta} \left(\; \tau_{\geq i}(MU \otimes X) \rightrightarrows \tau_{\geq i}(MU \otimes MU \otimes X) \dots . \; \right)$$

leading to a spectral sequence

$$\mathrm{Ext}_{MU_*MU}(MU_*, MU_*X) \Rightarrow \pi_*(X),$$

called the Adams-Novikov spectral sequence.

In fact, one can categorify this:

Definition 2.5.

$$\mathrm{Syn}_{MU} := \lim_{\Delta}^{\mathrm{Pr}^L_\omega} \mathrm{Mod}_{\tau_{\geq *}(MU^{\otimes \bullet+1}\otimes X)}(\mathrm{FilSp}) \simeq \mathrm{Ind}(\lim_{\Delta} \mathrm{Mod}_{\tau^{\omega}_{\geq *}(MU^{\otimes \bullet+1}\otimes X)}(\mathrm{FilSp}))$$

This is a presentable stable ∞-category. We have a functor $\nu : \mathrm{Sp} \to \mathrm{Syn}_{MU}$ arising from the functors $X \mapsto \tau^{\geq *}(MU^{\otimes \bullet+1}) \otimes X$. One also has a forgetful functor to FilSp, as the Ind-extension of the functor that takes a diagram $X_\bullet$, forgets to FilSp, and takes the limit. So an object of Syn_{MU} has an underlying object in FilSp (and, further, an "underlying spectrum" in Sp obtained by taking the colimit). From the definitions, the underlying filtered spectrum of $\nu(X)$ leads to the Adams-Novikov spectral sequence of X, so general objects in Syn_{MU} can be thought of as "Adams-Novikov-like filtered spectra". In fact, $\mathrm{Syn}_{MU} \simeq \mathrm{Mod}_{\nu(\mathbb{S})}(\mathrm{FilSp})$, and so if we write

$$\mathbb{S}^{n,w} := \Sigma^n \nu(\mathbb{S})(w),$$

with (w) denoting a filtration shift, objects in Syn_{MU} can be built from "cells" $\mathbb{S}^{n,w}$. Such a cell has underlying filtered spectrum with associated spectral sequence a copy of the Adams-Novikov spectral sequence of the sphere, but shifted to be starting in degree $(n, w-n)$ instead of $(0,0)$. More generally, we can write $\Sigma^{n,w}X$ for a combination of suspension and filtration-shift by w. We can also write $\pi_{n,w}(X) = [\mathbb{S}^{n,w}, X]$.

In terms of these bigraded homotopy groups, we can actually give a nice "black-box" way of thinking about the Adams-Novikov spectral sequence corresponding to a synthetic spectrum. If we write $\tau : \mathbb{S}^{0,-1} \to \mathbb{S}^{0,0}$ for the map coming from the filtration structure map, $X/\tau = X \otimes (\mathbb{S}/\tau)$ is levelwise given by the associated graded terms. So $\pi_{*,*}(X/\tau)$ is the E_2-page. The bigraded homotopy groups $\pi_{*,*}(X)$ relate to these by a "τ-Bockstein spectral sequence", and the homotopy groups of the underlying spectrum are obtained from those by taking a colimit.

Having the fundamental object not be spectral sequences but filtered spectra allows us to construct interesting new spectral sequences (e.g. by taking cofibers). Additionally, the freedom coming from shifting in the filtration direction w allows us to build objects which represent Adams-Novikov spectral sequences with modified filtrations. This idea of "modified Adams filtrations" turns out to be extremely useful, and predates synthetic spectra; to my knowledge it originated in work of Bob Bruner, who used it to analyze power operations in the Adams spectral sequence.

To illustrate the utility of these modified Adams(-Novikov) spectral sequences, I want to use this to present a proof of the Nilpotence Theorem due to

Robert Burklund (so far unpublished). It is based on the following observation from my PhD thesis:

Theorem 2.6. *If $X \in (\mathrm{Syn}_{MU})^\omega$, the bigraded homotopy groups $\pi_{n,w}(X)$ admit a minimal vanishing line, whose slope is (in the $(n, w-n)$ grading) of the form*

$$\mathrm{slope}(X) = \frac{1}{(p^i - 1)p^{j+1} - 1}$$

for some prime p, $i \geq 1$ and $j \geq 0$.

Additionally, X admits a self-map $v : \Sigma^{n,w} X \to X$ with $n > 0$, $\frac{w-n}{n} =$ $\mathrm{slope}(X)$, and such that the cofiber X/v has a vanishing line of slope strictly smaller than $\mathrm{slope}(X)$.

This is proved first proving an algebraic version of this statement (for derived MU_*MU comodules), by an inductive argument over quotients of MU_*MU. Using that the derived MU_*MU-statement can be interpreted as a statement mod τ, vanishing lines and self-maps can then be lifted through the τ-Bockstein spectral sequence.

Robert makes the following observation from this:

Lemma 2.7 (Burklund)**.** *There exists a sequence of $C_i \in (\mathrm{Syn}_{MU})^\omega$ with maps $\mathbb{S}^{0,0} \to C_i$ with the following properties:*

1. *The slope of the minimal vanishing line* $\mathrm{slope}(C_i)$ *is decreasing, and converges to 0 as $i \to \infty$.*

2. *The image of C_i in* Sp *splits as a nonzero sum of finitely many spheres, with $\mathbb{S}^{0,0} \to C_i$ inducing the inclusion of one of the summands.*

Proof. We begin with $C_0 = \mathbb{S}^{0,0}$. Inductively, by Theorem 2.6, C_i has a minimal vanishing line (slope 1 for $i = 0$), and a self-map v whose cofiber has an even lower vanishing line. So $C_{i+1} = C_i/v$ would already have a lower vanishing line. The statement about the slopes converging to 0 is automatic in any case, since the set of possible slopes does not have other accumulation points than 0. What about the last condition? The underlying spectrum of C_i is a sum of spheres. So the map induced by v on the underlying spectrum is a matrix with entries given by elements of $\pi_*(\mathbb{S})$. By replacing v by a suitable power, we may assume its degree to be large enough that all these entries have positive degree, and then further that they are 0, since by Nishida's special case of the nilpotence theorem all positive-degree elements of $\pi_*(\mathbb{S})$ are nilpotent. If we replace v by such a power, the cofiber will still have underlying spectrum a sum of spheres. $\square$

It turns out this is all one needs to prove the Nilpotence Theorem, a celebrated result originally due to Devinatz-Hopkins-Smith:

Theorem 2.8 (Nilpotence Theorem)**.** *Assume R is a connective ring spectrum, and $\alpha \in \pi_*(R)$ an element whose image in $MU_*(R)$ is zero. Then there exists k with $\alpha^k = 0$.*

(There are various other ways to state this, for example as a statement about maps $f : X \to Y$ between finite spectra being smash-nilpotent if they are nullhomotopic after tensoring with MU. All of these follow from the above by easy arguments using Spanier-Whitehead duals.)

Burklund's proof of the Nilpotence Theorem. α being zero in $MU_*(R)$ means that it is detected above the 0-line in the ANSS of R. This means that it refines to a map $\widetilde{\alpha} : \mathbb{S}^{n,n+1} \to \nu(R)$ of slope $\frac{1}{n}$. If $\nu(R)$ already had a vanishing line of slope $< \frac{1}{n}$, some power $\widetilde{\alpha}^k$ would be 0 for degree reasons. By the above Lemma, we can find C_i with slope$(C_i) < \frac{1}{n}$. Now (using an argument about a cell decomposition of $\nu(R)$), we have that $\nu(R) \otimes C_i$ has a vanishing line of slope $< \frac{1}{n}$, so some power $\widetilde{\alpha}^k$ is zero in $\nu(R) \otimes C_i$. But since after passing to underlying spectra, the map $\nu(R) \to \nu(R) \otimes C_i$ splits, it means that $\alpha^k \in \pi_*(R)$ is zero. □

3 ~~Talk 3~~Bonus content: The effective spectra project

(Joint with Ben Antieau)

In some other parts of math (e.g. algebra and number theory), there are some impressive computer algebra systems and databases (for example, see `https://lmfdb.org`).

In homotopy theory, we also have a strong culture of computations, but comparatively fewer computer algebra projects. There are individual projects that make impressive use of machine computation, and some specialized tools (e.g. from the Adams spectral sequence people), but nothing that feels quite as general as some of the computer algebra tools and databases available in other parts of mathematics.

Claim: The main reason for this is that in those areas, the basic objects (if they are suitably finite) are straightforward to represent in a computer (linear algebra stuff by dimensions and matrices, finite groups by permutations or matrix representations, finite-type algebras by generators and ideals, number fields by polynomials,...), while in a field like stable homotopy theory, even some of the most basic objects are surprisingly hard to encode!

The type of representation/encoding we are looking for needs to satisfy several criteria. Obviously, it has to be finite, but also it should be clear how to pass from the representation to the abstract mathematical object, and ideally we should be able to decide various questions algorithmically from it. For example, given representations of two different objects, we would want to be able to decide if they are equivalent. We will refer to such a representation as "(computationally) effective".

For example, let us think about how to represent finite spectra. A finite spectrum admits a cell structure with finitely many cells. Inductively, we could describe it for example in terms of its $(n-1)$-skeleton X_{n-1} and homotopy classes of attaching maps $\varphi_\alpha \in \pi_{n-1}(X_{n-1})$. Since X_{n-1} is finite, that group is finitely generated, so in principle, the attaching maps can be described by finite data, for example as linear combination of finitely many generators.

The obvious problem here is that to even describe X_n, we already need to have computed $\pi_{n-1}(X_n)$. And it doesn't even suffice to know its isomorphism type: If I say "the spectrum with a 0-cell and a 9-cell attached along $\varepsilon + \eta\sigma$", to know which one I am referring to you really need to know more than just $\pi_8(\mathbb{S}) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$, you really need to know which elements I mean. For complexes consisting of two cells like that, the information lives completely in $\pi_*(\mathbb{S})$, for which we do have quite a large range computed, and agreed-upon standard names for generators in a range. But if we attach more cells, the necessary information becomes unwieldy: To describe 3-cell complexes, you would need to have some table like that for all 2-cell complexes. Alternatively (e.g. through this formalism of coherent chain complexes applied to the skeletal filtration), one can also think of a 3-cell complex as involving a pair of attaching maps in $\pi_*(\mathbb{S})$ between "adjacent" cells, as well as a nullhomotopy of their composite. This runs into the following issue: How many different homotopy classes of homotopies $2 \circ \eta \simeq 0$ are there? (Answer: 2, since they are paths in $\mathrm{Map}(\mathbb{S}^1, \mathbb{S}^0)$, whose π_1 is $\pi_2(\mathbb{S}) \cong \mathbb{Z}/2$.) But again, it seems difficult to encode which one we choose.

One unambiguous way to represent a finite spectrum comes from the following observation, which is for example easy to see from Freudenthal stability and some standard result that every finite CW complex has the homotopy type of a finite simplicial complex:

Proposition 3.1. *Every finite spectrum has the form $\Sigma^{\infty-n}K$ for some finite simplicial complex K.*

This does technically allow us to encode a finite spectrum unambiguously (since K can be encoded purely combinatorially, and n is just a number), but it isn't very practical, and there is an annoying problem: I could represent for example $\mathrm{cofib}(\eta : \mathbb{S}^1 \to \mathbb{S}^0)$ as $\Sigma^{\infty-2}\,\mathrm{cofib}(\eta : S^3 \to S^2)$ and pick some small triangulation of $\mathrm{cofib}(\eta : S^3 \to S^2) \simeq \mathbb{C}P^2$, but if I instead represented it by the unstable $\mathrm{cofib}(\Sigma^{100}\eta : S^{103} \to S^{102})$, I will end up with many many more simplices! This comes from the fact that an n-simplex has to the order of n^k many codimension k simplices, so the combinatorial complexity grows quite quickly with the dimension. This is in stark contrast that we know homotopically that things should become easier as we suspend, and then stabilize: $\pi_3(S^2) \cong \mathbb{Z}$, but $\pi_{k+3}(S^{k+2}) \cong \mathbb{Z}/2$ for any $k \geq 1$. So the usual ways to encode unstable homotopy types simplicially just do not know about stability phenomena!

Instead, we should look for inherently stable ways to describe a spectrum through algebraic data. One good starting point is the Adams descent (from talk 2) for p-complete bounded below X:

$$X \simeq \lim_{\Delta}(H\mathbb{F}_p^{\otimes \bullet+1} \otimes X)$$

Each of the objects in the diagram is an $H\mathbb{F}_p$-module (picking one of the tensor factors), and $\mathrm{Mod}_{H\mathbb{F}_p}(\mathrm{Sp}) \simeq \mathcal{D}(\mathbb{F}_p)$. We know how to model those objects algebraically in an inherently stable way, by chain complexes!

However, the structure maps in the diagram (and the homotopies relating those: Remember, a Δ-shaped diagram in Sp is really a map of simplicial sets $N(\Delta) \to \mathrm{Sp}$, and there are a lot of simplices in this nerve in addition to just the objects and morphisms) are not all $H\mathbb{F}_p$-linear (for the same choice of $H\mathbb{F}_p$). This issue came not up when setting up the Adams spectral sequence: When we pass to homotopy groups, the two $\mathbb{F}_p$-module structures on $\pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$ are the same. But here we are trying to say something more coherently.

So it is hard to say where this diagram should live besides Sp. There is a different perspective on essentially the same information, that seems more useful:

Proposition 3.2. *Write* $\mathrm{Sp}_{p,+}^{\wedge}$ *for* p*-complete bounded below spectra, and* $\mathcal{D}(\mathbb{F}_p)_+$ *for (homologically) bounded below objects in the derived category. Then*

$$\mathrm{Sp}_{p,+}^{\wedge} \rightleftarrows \mathcal{D}(\mathbb{F}_p)_+$$

is monadic *and comonadic.*

The monadicity is the usual statement that $\mathcal{D}(\mathbb{F}_p)$ can be interpreted as modules in Sp (with or without p-completion, and with or without boundedness). But the comonadicity tells us something interesting: There is a comonad $Q_{\mathbb{F}_p}$ on $\mathcal{D}(\mathbb{F}_p)_+$, for which

$$\mathrm{Sp}_{p,+}^{\wedge} \simeq \mathrm{coAlg}_{Q_{\mathbb{F}_p}}(\mathcal{D}(\mathbb{F}_p)_+),$$

describing spectra as "chain complexes with extra structure"!

This comonad has underlying functor $H\mathbb{F}_p \otimes_{\mathbb{S}} -$, and can be described (in terms of Morita theory) by the bimodule $H\mathbb{F}_p \otimes_{\mathbb{S}} H\mathbb{F}_p$. We can describe $\mathcal{D}(\mathbb{F}_p)$ in terms of chain complexes, so we could hope to represent $Q_{\mathbb{F}_p}$ explicitly on chain complexes. However, we can't hope for something too strict: Anything too chain-complexy (i.e. dg categories) gives $\mathbb{Z}$-linear ∞-categories and functors, but $Q_{\mathbb{F}_p}$ can't be $\mathbb{Z}$-linear (since for example the coalgebras – spectra – aren't).

The underlying functor of $Q_{\mathbb{F}_p}$ has a description due to Eilenberg-MacLane, at least its restriction to the full subcategory $\mathrm{Vect}_{\mathbb{F}_p} \subseteq \mathcal{D}(\mathbb{F}_p)$, taking V to a big complex whose degree k-entry is some quotient of $\mathbb{F}_p[V^{2^k}]$. It is not additive, only additive up to chain homotopy, solving the apparent paradox.

There is another solution, which is to take ∞-categories more seriously. Instead of trying to strictify $Q_{\mathbb{F}_p}$ to a functor on the 1-categories, we could directly think about the ∞-categories as simplicial sets. For example, $\mathcal{D}(\mathbb{F}_p)$ can be described as $N^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{F}_p))$, the dg-nerve. Here, 0-simplices are chain complexes, 1-simplices are morphisms, 2-simplices are (f, g, h, H) with H a homotopy $g \circ f \simeq h$, etc. A functor is then just an assignment taking n-simplices to n-simplices, but it does not have to strictly respect composition: Instead, it could for example take a "strict" 2-simplex coming from $(f, g, g \circ f, 0)$ to a 2-simplex $(F(f), F(g), F(g \circ f), H)$ with an interesting homotopy. And even if $F(g \circ f) = F(g) \circ F(f)$, it can still produce an interesting homotopy (which will then be a degree +1 chain map). So our ∞-categorically additive functor cannot be modeled by an additive functor on chain complexes, but it turns out it can be modeled by one of these assignments on simplices whose effect on 0 and 1-simplices looks additive, but then there are interesting nonlinear effects on the chain homotopies on 2-simplices, and similarly on higher simplices.

To emphasize this point of view, let us call a map of simplicial sets $N^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{F}_p)) \to N^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{F}_p))$ an "∞-functor" (of course, this is the only thing what a functor between ∞-categories could mean). When we describe a functor $\mathcal{D}(\mathbb{F}_p) \to \mathcal{D}(\mathbb{F}_p)$ as derived functor of a 1-categorical functor $\mathrm{Ch}(\mathbb{F}_p) \to \mathrm{Ch}(\mathbb{F}_p)$, we essentially describe it as an ∞-functor with trivial coherence homotopies, and only have to describe what happens on 0-simplices (objects) and 1-simplices (morphisms). If we instead relax this and allow ourselves to work with general ∞-functors, the price we pay is that we have to describe the effect on n-simplices for all n now, but we gain added flexibility. For example, we can change the effect of a functor F on, say, the 1-skeleton, by a kind of homotopy extension. This allows us to find smaller representatives of the functor $Q_{\mathbb{F}_p}$. Currently, we have explicit formulas worked out for low-dimensional simplices, which already behave in an interesting way:

Proposition 3.3 (K.-Antieau)**.** *The functor $Q_{\mathbb{F}_p} : \mathcal{D}(\mathbb{F}_p) \to \mathcal{D}(\mathbb{F}_p)$ is equivalent to one which, on $N^{\mathrm{dg}}(\mathrm{Ch}(\mathbb{F}_p))$, acts as following on low-dimensional simplices:*

- *On 0- and 1-simplices, it acts the same as $\mathcal{A}_* \otimes -$, where $\mathcal{A}_* = \pi_*(H\mathbb{F}_p \otimes H\mathbb{F}_p)$ is viewed as chain complex with the zero differential.*
- *On 2-simplices, its effect on the chain homotopy involves an interesting quadratic formula (in the matrix entries of the chain maps).*

We are working on pushing these formulas both to higher-dimensional simplices and to also include the comonad structure (the natural transformation $Q \to Q \circ Q$, the coassociativity homotopy between the resulting two natural transformations $Q \to Q^{\circ 3}$, etc.) One way to do this is using explicit chain homotopy equivalences $\mathcal{A}_* \otimes M \simeq Q_{\mathbb{F}_p}(M)$ to essentially transport the structure from the strict functorial description (from Eilenberg-MacLane). The effect on higher-dimensional simplices arises from iterating the maps representing the chain homotopies.

Suppose we have a description of not just the underlying functor of $Q_{\mathbb{F}_p}$ but also the comonad structure, in terms of explicit formulas on the n-simplices of the dg nerve. What is a coalgebra in those terms? It has an underlying chain complex M, and a coaction map $M \to Q(M)$ (with a counitality condition). Instead of the usual coassociativity condition, part of the data is a homotopy filling the square

$$\begin{tikzcd} M \arrow[r, "\mathrm{coact}"] \arrow[d, "\mathrm{coact}"] & Q(M) \arrow[d, "Q(\mathrm{coact})"] \\ Q(M) \arrow[r, "\Delta_Q"] & Q(Q(M)) \end{tikzcd}$$

using the coaction and the comonad structure map (and a counitality condition). One level higher, we have a homotopy of homotopies filling a cube-shaped diagram with initial vertex M and terminal vertex $Q(Q(Q(M)))$, and faces obtained in different ways, some of them by applying Q to the above square. This is where the nonlinear effect of Q on 2-simplices starts to contribute.

We have some evidence that the action on higher-dimensional simplices also admits polynomial descriptions (meaning polynomial formulas in the matrix entries). So there is hope for some kind of closed-form description: After all, on homotopy groups it is given by the dual Steenrod algebra, which we understand fully. Having that, we could really encode a p-complete spectrum by a chain complex M with a sequence of maps $M \to Q^{\circ n}(M)[1-n]$ representing chain homotopies with some nonlinear description for their boundary. For each n, if the source is finite-dimensional, the target is finite-type, and so that is finite information, but of course it is still an infinite sequence. We are also exploring a $\mathbb{Z}$-based version instead, where one has an additional connectivity statement which would really lead to finite data.

In a slightly different direction, we could just work with this data "up to n-dimensional cells", i.e. stop at the homotopy corresponding to an n-dimensional cube (plus a cocycle condition coming from the boundary of the $n+1$-dimensional cube). This corresponds to only looking at

$$\mathrm{coAlg}_Q(\mathrm{ho}_n(\mathcal{D}(\mathbb{F}_p))) =: n\,\mathrm{Comod}_{\mathcal{A}_*}$$

This only involves the action of Q on simplices up to dimension n. It does not quite give us spectra, but knows about some of the higher information in the Adams spectral sequence. Morally, this is because for a spectrum X, the full Q-coalgebra structure on $H\mathbb{F}_p \otimes X$ contains the same information as the cosimplicial Adams resolution. Passing to homotopy n-categories is closely related to applying the truncations $\tau_{[i,i+n-1]}$, which as we saw in the second talk gives the $F^{[i,i+n-1]}$-truncations of the Adams filtration. So this contains information about the differentials up to d_n. (In the language of $\mathbb{F}_p$-synthetic spectra, this is saying that these n-ary comodules can be used to compute $\nu_{\mathbb{F}_p}(X)/\tau^n$, which is a filtered spectrum whose i-th term is exactly the $F^{[i,i+n-1]}(X)$.)

In the $n = 1$ case, this is just the classical comodule story. In the $n = 2$ case, the dual picture has been developed by Baues as the "secondary Steenrod algebra", and used by Baues-Jibladze to describe the Adams E_3 page completely algebraically, with later improvements by Nassau and Chua.

The $n = 2$ descriptions are in terms of some differential graded algebra. There is a crucial issue with extending that approach to $n \geq 3$: For $n = 1$ and $n = 2$ (and $p = 2$), the truncations $F^{[0,n-1]}(X)$ of the Adams filtration are given by $H\mathbb{F}_2$ and $H\mathbb{Z}/4$. For $n = 3$ however, the corresponding truncation has $\pi_0 \cong \mathbb{Z}/8$ and $\pi_1 \cong \mathbb{Z}/2$, generated by η. In particular, it cannot be a $H\mathbb{Z}$-module. This provides an obstruction to a dga-like description. In our approach (where we first try to reproduce the secondary Steenrod algebra results, and then extend to $n \geq 3$), this issue does not occur, at the price of introducing these nonlinear effects on higher simplices.